APPENDIX A

Determination of the Degradation of Analyzer Sensitivity and Resolution as a Function of Sweep Rate

Let us make the assumption that the analyzer selectivity curve is given by a Gaussian error function. This is true to a good approximation for an i-f amplifier composed of several synchronously-tuned stages.

The steady-state voltage amplitude response (normalized) is then

$$H(\omega) \doteq \exp\left(\frac{-\ln 2}{2\pi^2}\right)\left(\frac{\omega - \omega_o}{B}\right)^2 \quad (1)$$

where B is the 3-db bandwidth in cps and ω_o is the angular center frequency (see figure A1).

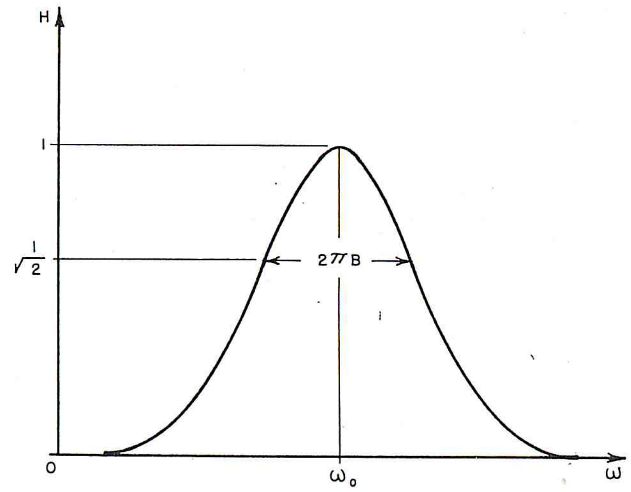

Figure A1. Normalized Gaussian Amplitude Response Curve Representing Selectivity of Analyzer I-F Amplifier.

The swept signal voltage may be represented by

$$e(t) = \epsilon^{j\theta} \quad (2)$$

where θ is the instantaneous phase angle; hence the instantaneous angular frequency is given by

$$\omega_i = \frac{d\theta}{dt} \quad (3)$$

Now, assuming a linearly swept signal as shown in figure A2, where $\frac{F}{T}$

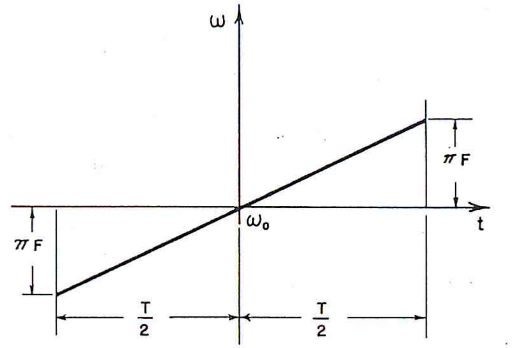

Figure A2. Time Variation of Swept Signal Frequency.

is the sweep rate in cycles per second per second, and $\omega = \omega_o$ at $t = 0$, we have for ω_i;

$$\omega_i = \omega_o + \frac{2\pi F}{T} t \tag{4}$$

Therefore, from equation (3),

$$\theta = \int \omega_i \, dt = \omega_o t + \frac{\pi F}{T} t^2 \tag{5}$$

Substituting into equation (2);

$$e(t) = \exp j \left(\omega_o t + \frac{\pi F}{T} t^2 \right) \tag{6}$$

The frequency spectrum of $e(t)$ is given by the Fourier transform

$$E(\omega) = \int_{-\infty}^{\infty} e(t) \, \epsilon^{-j\omega t} \, dt \tag{7}$$

$$= \int_{-\infty}^{\infty} \epsilon^{j[(\omega_o - \omega) t + \frac{\pi F}{T} t^2]} \, dt \tag{8}$$

Equation (8) may be evaluated as follows: Consider the integral

$$\int_{-\infty}^{\infty} \epsilon^{j(at^2 + bt)} \, dt \tag{9}$$

Completing the square in the exponential,

$$at^2 + bt = a\left(t + \frac{b}{2a} \right)^2 - \frac{b^2}{4a} \tag{10}$$

Let

$$x = t + \frac{b}{2a} \tag{11}$$

whence

$$dx = dt \tag{12}$$

Substituting in equation (9) we obtain

$$\int_{-\infty}^{\infty} \epsilon^{j(at^2 + bt)} \, dt = \epsilon^{-j\frac{b^2}{4a}} \int_{-\infty}^{\infty} \epsilon^{jax^2} \, dx = 2\epsilon^{-j\frac{b^2}{4a}} \int_{0}^{\infty} \epsilon^{jax^2} \, dx \tag{13}$$

Now, equation (13) is a definite integral which may be readily evaluated by applying No. 492 of Pierce's Short Table of Integrals, viz.

$$\int_0^{\infty} \epsilon^{-a^2 x^2} \, dx = \frac{1}{2a} \sqrt{\pi} \tag{14}$$

Thus, equation (13) becomes

$$\int_{-\infty}^{\infty} \epsilon^{j(at^2 + bt)} \, dt = \sqrt{\frac{j\pi}{a}} \, \epsilon^{-j\frac{b^2}{4a}} \tag{15}$$

Making the appropriate substitutions in equation (8), we obtain

$$E(\omega) = \sqrt{\frac{jT}{F}} \exp - j \frac{T}{4\pi F} (\omega_o - \omega)^2 \tag{16}$$

The output voltage transient of the Gaussian amplifier is given by the inverse Fourier transform

$$V(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(\omega) \, H(\omega) \, \epsilon^{j\omega t} \, d\omega \tag{17}$$

$$= \frac{1}{2\pi} \sqrt{\frac{jT}{F}} \int_{-\infty}^{\infty} \epsilon^{-j\frac{T}{4\pi F}(\omega_o - \omega)^2} \, \epsilon^{-\left(\frac{\ln 2}{2\pi^2 B^2}\right)(\omega - \omega_o)^2} \, \epsilon^{j\omega t} \, d\omega \tag{18}$$

The evaluation of equation (18) may be simplified by making the following substitutions: Let

$$c = \frac{T}{4\pi F} \tag{19}$$

and

$$d = \frac{\ln 2}{2\pi^2 B^2} \tag{20}$$

also let

$$K = \frac{1}{2\pi}\sqrt{\frac{jT}{F}} \tag{21}$$

then equation (18) becomes

$$V(t) = K\int_{-\infty}^{\infty} \epsilon^{-d(\omega-\omega_o)^2}\, \epsilon^{-jc(\omega_o-\omega)^2}\, \epsilon^{j\omega t}\, d\omega \tag{22}$$

$$= K\epsilon^{-\omega_o^2(d+jc)} \int_{-\infty}^{\infty} \epsilon^{\left\{-(d+jc)\omega^2 + [2d\omega_o + j(2c\omega_o + t)]\omega\right\}} d\omega \tag{23}$$

Let

$$Z_1 = d + jc \tag{24}$$

and

$$Z_2 = 2d\omega_o + j\,(2c\,\omega_o + t) \tag{25}$$

Substituting in equation (23), we obtain

$$V(t) = K\epsilon^{-\omega_o^2 Z_1} \int_{-\infty}^{\infty} \epsilon^{-Z_1\omega^2 + Z_2\omega}\, d\omega \tag{26}$$

Now

$$-Z_1\omega^2 + Z_2\omega = -Z_1\left(\omega - \frac{Z_2}{2Z_1}\right)^2 + \frac{Z_2^2}{4Z_1} \tag{27}$$

hence

$$V(t) = K\epsilon^{-\omega_o^2 Z_1}\, \epsilon^{\frac{Z_2^2}{4Z_1}} \int_{-\infty}^{\infty} \epsilon^{-Z_1\left(\omega - \frac{Z_2}{2Z_1}\right)^2} d\omega \tag{28}$$

Now let

$$W = \omega - \frac{Z_2}{2Z_1} \tag{29}$$

whence

$$dW = d\omega \tag{30}$$

$$\therefore V(t) = K\epsilon^{-\omega_o^2 Z_1}\, \epsilon^{\frac{Z_2^2}{4Z_1}} \int_{-\infty}^{\infty} \epsilon^{-Z_1 W^2}\, dW \tag{31}$$

Again applying equation (14), we obtain for the integral factor of equation (31)

$$\int_{-\infty}^{\infty} \epsilon^{-Z_1 W^2}\, dW = 2\int_0^{\infty} \epsilon^{-Z_1 W^2}\, dW = \sqrt{\frac{\pi}{Z_1}} \tag{32}$$

Thus, we have for equation (31),

$$V(t) = K\sqrt{\frac{\pi}{Z_1}}\, \epsilon^{-\omega_o^2 Z_1}\, \epsilon^{\frac{Z_2^2}{4Z_1}} \tag{33}$$

$$= \frac{1}{2\pi}\sqrt{\frac{j\pi T}{F(d+jc)}}\, \exp\left\{\frac{[2d\,\omega_o + j\,(2c\,\omega_o + t)]^2}{4\,(d+jc)} - \omega_o^2\,(d+jc)\right\} \tag{34}$$

Now, it must be remembered that we are actually interested in determining the envelope, $A(t)$, of the real part of $V(t)$. Since $V(t)$ is complex in our case, it may be represented by

$$V(t) = |V(t)|\,\epsilon^{j\phi(t)} \tag{35}$$

$$\therefore Re\,\{\,V(t)\,\} = |\,V(t)\,|\cos\phi\,(t) \tag{36}$$

which has the envelope

$$A(t) = |V(t)| \tag{37}$$

Now equation (34) has the form

$$V(t) = Z\exp\,[\phi_1(t) + j\,\phi_2(t)] \tag{38}$$

where Z is complex, and $\phi_1(t)$ and $\phi_2(t)$ are real. Hence, from equation (37),

$$A(t) = |\,Z\,|\exp\phi_1(t) \tag{39}$$

Now

$$Z = \frac{1}{2\pi}\sqrt{\frac{\pi T}{F}}\cdot\sqrt{\frac{j}{d+jc}} \tag{40}$$

$$\therefore |Z| = \frac{1}{2\pi}\sqrt{\frac{\pi T}{F}}\,(c^2 + d^2)^{-1/4} \tag{41}$$

also

$$\phi_1(t) + j\,\phi_2(t) = \frac{[2d\,\omega_o + j\,(2c\,\omega_o + t)]^2}{4\,(d+jc)} - \omega_o^2\,(d+jc) \tag{42}$$

whence

$$\phi_1(t) = \frac{-dt^2}{4(c^2+d^2)} \qquad (43)$$

Substituting equations (41) and (43) into equation (39), we obtain

$$A(t) = \frac{1}{2\pi}\sqrt{\frac{\pi T}{F}}\,(c^2+d^2)^{-1/4} \;\exp \frac{-d}{4(c^2+d^2)}\,t^2 \qquad (44)$$

It will be noted that $A(t)$ has the form of the Gaussian error function. Now let a be the maximum amplitude of $A(t)$. Thus

$$a = \frac{1}{2\pi}\sqrt{\frac{\pi T}{F}}\,(c^2+d^2)^{-1/4} \qquad (45)$$

Substituting equations (19) and (20) into equation (45), we obtain, for a

$$a = \left[1 + \left(\frac{2\ln 2}{\pi}\right)^2 \left(\frac{F}{TB^2}\right)^2\right]^{-1/4} \qquad (46)$$

$$= \left[1 + 0.195\left(\frac{F}{TB^2}\right)^2\right]^{-1/4} \qquad (47)$$

The resolution of the spectrum analyzer is the apparent bandwidth of the output transient as viewed on the screen. This is given by

$$R = \frac{F}{T} \cdot \Delta t \qquad (48)$$

where Δt is the time required to sweep between the 3-db points of the transient envelope. Referring to equation (44), it follows that

$$\frac{1}{\sqrt{2}} = \exp \frac{-d}{4(c^2+d^2)}\left(\frac{\Delta t}{2}\right)^2 \qquad (49)$$

whence

$$\Delta t = \left[(8\ln 2)\left(\frac{c^2+d^2}{d}\right)\right]^{1/2} \qquad (50)$$

Substituting equations (19) and (20) into equation (50) yields

$$\Delta t = 4\pi B\left[\left(\frac{T}{4\pi F}\right)^2 + \left(\frac{\ln 2}{2\pi^2 B^2}\right)^2\right]^{1/2} \qquad (51)$$

Therefore, from equation (48),

$$\frac{R}{B} = \left[1 + \left(\frac{2\ln 2}{\pi}\right)^2 \left(\frac{F}{TB^2}\right)^2\right]^{1/2} \qquad (52)$$

$$= \left[1 + 0.195\left(\frac{F}{TB^2}\right)^2\right]^{1/2} \qquad (53)$$